

n -ary associative algebras, cohomology, free algebras and coalgebras

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Abstract

When n is odd, a cohomology of type Hochschild for n -ary partially associative algebras has been defined in Gnedbaye's thesis. Unfortunately, the cohomology definition is not valid when n is even. This fact is found again in the computations of the n -ary partially associative free algebra.

In this work, we define in a first time two approaches of an Hochschild cohomology for n -ary partially associative algebras. First by reducing the space of cochains, secondly by using a graded version. Next we compute the free n -ary algebra, giving a basis of this algebra. At last we extend the notion of coalgebras to n -ary algebras.

All algebraic objects will be considered over a commutative field \mathbb{K} of characteristic zero.

1 Relations between n -ary partially associative algebras and Gerstenhaber products

1.1 Definition

Let V be a \mathbb{K} -vector space and consider

$$C^k(V) = \text{Hom}_{\mathbb{K}}(V^{\otimes k}, V),$$

for any natural number k . By definition a n -ary partially associative algebra is a pair (V, μ) where V is a \mathbb{K} -vector space and μ a linear map $\mu : V^{\otimes n} \rightarrow V$ satisfying

$$\sum_{i=1}^n (-1)^{(i-1)(n-1)} \mu(X_1, \dots, \mu(X_i, \dots, X_{i+n-1}), X_{i+n-1}), \dots, X_{2n-1}) = 0. \quad (1)$$

When

$$\mu(X_1, \dots, \mu(X_i, \dots, X_{i+n-1}), \dots, X_{2n-1}) = \mu(X_1, \dots, \mu(X_j, \dots, X_{j+n-1}), \dots, X_{2n-1})$$

for any $i, j \in \{1, \dots, p\}$, the algebra (V, μ) is totally associative.

1.2 Gerstenhaber products $\bullet_{n,n}$

These products have been proposed by Gerstenhaber in the study of spaces of Hochschild cohomology of an associative algebra. We recall this quickly in order to use the practical notations which appear in the work of Gerstenhaber.

The Gerstenhaber product of $f \in C^n(V)$ and $g \in C^m(V)$ is the element $f \bullet_{n,m} g \in C^{n+m}(V)$ defined by

$$f \bullet_{n,m} g(X_1 \otimes \cdots \otimes X_{n+m-1}) = \sum_{i=1}^n (-1)^{(i-1)(m-1)} f(X_1 \otimes \cdots \otimes g(X_i \otimes \cdots \otimes X_{i+m-1}) \otimes \cdots \otimes X_{n+m-1}).$$

These Gerstenhaber products satisfy pre-Lie Identity [see Ge] that is:

$$(f \bullet_{n,m} g) \bullet_{n+m-1,p} h - f \bullet_{n,m+p-1} (g \bullet_{m,p} h) = (-1)^{(m-1)(p-1)} ((f \bullet_{n,p} h) \bullet_{n+p-1,m} g - f \bullet_{n,n+p-1} (h \bullet_{p,m} g)),$$

for any $f \in C^n(V)$, $g \in C^m(V)$ and $h \in C^p(V)$.

Notations. We denote the products of Gerstenhaber by $\bullet_{n,k}$. When there is no confusion, we denote these products simply by \bullet . Moreover, the symbol \circ refers to the ordinary composition of applications.

Definition 1 We call n -ary algebra associated to $\bullet_{n,n}$ any \mathbb{K} -vector space V with an application $\mu \in C^n(V)$ satisfying:

$$\mu \bullet_{n,n} \mu = 0.$$

We denote it by $(V, \bullet_{n,n})$ or $(V, \bullet_{n,n}, \mu)$ if we need to specify the multiplication μ .

Then we have:

$$\mu \bullet_{n,n} \mu(X_1, \dots, X_{2n-1}) = \sum_{i=1}^n (-1)^{(i-1)(n-1)} \mu(X_1, \dots, \mu(X_i, \dots, X_{i+n-1}), \dots, X_{2n-1}) = 0. \quad (2)$$

These algebras correspond to partially associative algebras.

Remarks. We will study identities which are deduced from the definition of the product $\bullet_{n,n}$.

1. For $n = 1$, Identity (2) reduces to:

$$\mu \bullet_{1,1} \mu(X_1) = \mu(\mu(X_1)) = 0$$

so $\mu \circ \mu = 0$.

2. For $n = 2$ we get:

$$\mu \bullet_{2,2} \mu(X_1, X_2, X_3) = \mu(\mu(X_1, X_2), X_3) - \mu(X_1, \mu(X_2, X_3)) = 0$$

and $(V, \bullet_{2,2}, \mu)$ is an associative algebra.

3. For $n > 2$, the algebra $(V, \bullet_{n,n}, \mu)$ corresponds to a partially associative algebra (with operation in degree 0) studied by Gnedbaye (see [Gn]).

4. A n -ary algebra $(V, \bullet_{n,n}, \lambda)$ can not be deduced from an associative algebra $(V, \bullet_{2,2}, \mu)$ by composition that is λ can not be equal to

$$\sum_{k=1}^l a_k (Id_{p_{n-1}} \otimes \mu \otimes Id_{n-2-p_{n-1}}) \circ \cdots \circ (Id_{p_2} \otimes \mu \otimes Id_{n-2-p_2}) \circ (Id_{p_1} \otimes \mu \otimes Id_{n-p_1-2}).$$

5. A n -ary Lie algebra [Fi] is defined by a skew-symmetric product μ satisfying the generalized Jacobi Identity

$$\sum_{\sigma \in Sh_{n,n-1}} (-1)^{\varepsilon(\sigma)} \mu(\mu(X_{\sigma(1)}, \dots, X_{\sigma(n)}), X_{\sigma(n+1)}, \dots, X_{\sigma(2n-1)}) = 0$$

where $Sh_{n,n-1}$ is the set of $(n, n-1)$ -shuffles. If λ is a n -ary multiplication satisfying $\lambda \bullet_{n,n} \lambda = 0$, the product μ defined by

$$\mu(X_1, \dots, X_n) = \sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} \lambda(X_{\sigma(1)}, \dots, X_{\sigma(n)})$$

is a n -ary Lie algebra product.

Lemme 1 *Let $(V, \bullet_{n,n}, \mu)$ be a n -ary algebra with product $\bullet_{n,n}$. If n is even*

$$(\varphi \bullet \mu) \bullet \mu = 0,$$

for any $\varphi \in C^k(V)$.

Proof. It follows from pre-Lie identity that

$$(\varphi \bullet \mu) \bullet \mu - \varphi \bullet (\mu \bullet \mu) = (-1)^{(n-1)(n-1)} [(\varphi \bullet \mu) \bullet \mu - \varphi \bullet (\mu \bullet \mu)].$$

But $\mu \bullet \mu = 0$. Then, as n is even, we obtain

$$(\varphi \bullet \mu) \bullet \mu = -(\varphi \bullet \mu) \bullet \mu$$

and, using the fact that $char(\mathbb{K}) = 0$, this equation reduces to $(\varphi \bullet \mu) \bullet \mu = 0$.

Observe that for odd n , pre-Lie identity is trivial. In this case we compute $(\varphi \bullet \mu) \bullet \mu$. Let $\theta_k(\mu)$ be the map $V^{\otimes(2n+k-2)} \longrightarrow V^{\otimes k}$ defined by

$$\theta_k(\mu) = \sum_{\substack{0 \leq p \leq k-2 \\ 0 \leq q \leq k-2-p}} Id_p \otimes \mu \otimes Id_q \otimes \mu \otimes Id_{k-p-q-2}$$

where Id_0 means no operation, for example $Id_0 \otimes \mu \otimes Id_{k-2} \otimes \mu \otimes Id_0$ is just $\mu \otimes Id_{k-2} \otimes \mu$.

Lemme 2 *If n is odd, then for any $\varphi \in C^k(V)$*

$$(\varphi \bullet \mu) \bullet \mu = 2\varphi \circ \theta_k(\mu)$$

where \circ is the ordinary composition. In particular $(\varphi \bullet \mu) \bullet \mu = 0$ if and only if $Im \theta_k(\mu) \in Ker \varphi$.

Proof. The product \bullet satisfies

$$(\varphi \bullet \mu) \bullet \mu(X_1, \dots, X_{k+2n-2}) = \sum_{i=1}^{k+n-1} (\varphi \bullet \mu)(X_1, \dots, \mu(X_i, \dots, X_{i+n-1}), \dots, X_{k+2n-2}).$$

Terms of the right-hand side are of two kinds. The first corresponds of elements which can be rewritten as

$$\varphi(X_1, \dots, X_{p-1}, A, X_{2n+p-1}, \dots, X_{2n+k-2})$$

with $1 \leq p \leq k$ and

$$\begin{aligned} A &= \mu(\mu(X_p, \dots, X_{n+p-1}), X_{n+p}, \dots, X_{2n+p-2}) + \mu(X_p, \mu(X_{p+1}, \dots, X_{n+p}), \dots, X_{2n+p-2}) \\ &\quad + \dots + \mu(X_p, \dots, X_{n+p-2}, \mu(X_{n+p-1}, \dots, X_{2n+p-2})) \\ &= \mu \bullet \mu(X_p, \dots, X_{2n+p-2}) \\ &= 0. \end{aligned}$$

The second type corresponds to elements

$$\varphi(X_1, \dots, X_{p-1}, \mu(X_p, \dots, X_{n+p-1}), X_{n+p}, \dots, \mu(X_q, \dots, X_{n+q-1}), \dots, X_{k+2n-2})$$

with $1 \leq p \leq q - n \leq k - 1$. Then

$$\begin{aligned} (\varphi \bullet \mu) \bullet \mu(X_1, \dots, X_{k+2n-2}) &= \\ 2 \sum_{1 \leq p \leq q-n \leq k-1} \varphi(X_1, \dots, X_{p-1}, \mu(X_p, \dots, X_{n+p-1}), X_{n+p}, \dots, \mu(X_q, \dots, X_{n+q-1}), \dots, X_{k+2n-2}) &= \\ = 2\varphi(\theta_k(\mu))(X_1, \dots, X_{k+2n-2}). \end{aligned}$$

2 Cohomology of partially associative algebras $(V, \bullet_{n,n})$

Recall that if $n = 2$, the Hochschild cohomology of an associative algebra with multiplication μ is defined from the coboundary operator

$$\begin{aligned} \delta^k : C^k(V) &\longrightarrow C^{k+1}(V) \\ \delta^k \varphi &= (-1)^{k-1} \mu \bullet_{2,k} \varphi - \varphi \bullet_{k,2} \mu. \end{aligned}$$

Consider a n -ary algebra with a multiplication μ of type $\bullet_{n,n}$.

2.1 First case: n is even

Let φ be in $C^k(V)$. The applications $\mu \bullet \varphi$ and $\varphi \bullet \mu$ are in $C^{k+n-1}(V)$. We define, for any $i \in \{0, \dots, n-1\}$, the linear map:

$$\delta_i^k : C^{i+k(n-1)}(V) \longrightarrow C^{i+(k+1)(n-1)}(V)$$

by

$$\delta_i^k(\varphi) = (-1)^{k-1} \mu \bullet_{n,n+k-1} \varphi - \varphi \bullet_{n+k-1,n} \mu.$$

Theorem 1 *The maps δ_i^k satisfy*

$$\delta_i^{k+1} \circ \delta_i^k = 0,$$

for any $i = 0, 1, \dots, n-2$.

Proof. Consider

$$\begin{aligned} (\delta_i^{k+1} \circ \delta_i^k)(\varphi) &= (-1)^k \mu \bullet ((-1)^{k-1} \mu \bullet \varphi - \varphi \bullet \mu) - ((-1)^{k-1} \mu \bullet \varphi - \varphi \bullet \mu) \bullet \mu \\ &= -\mu \bullet (\mu \bullet \varphi) + (-1)^{k+1} \mu \bullet (\varphi \bullet \mu) + (-1)^k (\mu \bullet \varphi) \bullet \mu + (\varphi \bullet \mu) \bullet \mu. \end{aligned}$$

The pre-Lie identity implies that

$$(\mu \bullet \mu) \bullet \varphi - \mu \bullet (\mu \bullet \varphi) = (-1)^{(n-1)(k-1)} ((\mu \bullet \varphi) \bullet \mu - \mu \bullet (\varphi \bullet \mu)).$$

Since $\mu \bullet \mu = 0$ and n is even, we obtain

$$-\mu \bullet (\mu \bullet \varphi) = (-1)^{k-1} (\mu \bullet \varphi) \bullet \mu + (-1)^k \mu \bullet (\varphi \bullet \mu)$$

so

$$\begin{aligned} (\delta_i^{k+1} \circ \delta_i^k)(\varphi) &= ((-1)^{k-1} + (-1)^k) (\mu \bullet \varphi) \bullet \mu \\ &\quad + ((-1)^k + (-1)^{k+1}) \mu \bullet (\varphi \bullet \mu) + (\varphi \bullet \mu) \bullet \mu \\ &= (\varphi \bullet \mu) \bullet \mu. \end{aligned}$$

But, as n is even, Lemma 1 implies that $(\varphi \bullet \mu) \bullet \mu = 0$. So $\delta_i^{k+1} \circ \delta_i^k = 0$ and we have the following families of complexes

$$\begin{aligned} \mathcal{C}^0(V) &\xrightarrow{\delta_0^0} \mathcal{C}^{n-1}(V) \xrightarrow{\delta_0^1} \mathcal{C}^{2(n-1)}(V) \longrightarrow \dots \longrightarrow \mathcal{C}^{k(n-1)}(V) \xrightarrow{\delta_0^k} \mathcal{C}^{(k+1)(n-1)}(V) \longrightarrow \dots \\ \mathcal{C}^1(V) &\xrightarrow{\delta_1^0} \mathcal{C}^{n-1+1}(V) \xrightarrow{\delta_1^1} \mathcal{C}^{2(n-1)+1}(V) \longrightarrow \dots \longrightarrow \mathcal{C}^{k(n-1)+1}(V) \xrightarrow{\delta_1^k} \mathcal{C}^{(k+1)(n-1)+1}(V) \longrightarrow \dots \\ &\vdots \\ \mathcal{C}^i(V) &\xrightarrow{\delta_i^0} \mathcal{C}^{n-1+i}(V) \xrightarrow{\delta_i^1} \mathcal{C}^{2(n-1)+i}(V) \longrightarrow \dots \longrightarrow \mathcal{C}^{k(n-1)+i}(V) \xrightarrow{\delta_i^k} \mathcal{C}^{(k+1)(n-1)+i}(V) \longrightarrow \dots \\ &\vdots \\ \mathcal{C}^{n-2}(V) &\xrightarrow{\delta_{n-2}^0} \mathcal{C}^{2n-3}(V) \xrightarrow{\delta_{n-2}^1} \mathcal{C}^{3(n-1)-1}(V) \longrightarrow \dots \longrightarrow \mathcal{C}^{(1+k)(n-1)-1}(V) \xrightarrow{\delta_{n-2}^k} \mathcal{C}^{(k+2)(n-1)-1}(V) \longrightarrow \dots \end{aligned}$$

We can consider the associated cohomology.

2.2 Second case: n is odd

Consider a n -ary multiplication associated to Gerstenhaber's product with an odd n . Then pre-Lie Identity applied to triples (φ, μ, μ) with $\varphi \in \mathcal{C}^k(V)$ is always fulfilled. To define a cohomology for these algebras we have to restrict the space of cochains to the subspace $\chi^k(V)$ of k -linear applications $\varphi : V^{\otimes k} \longrightarrow V$ subject to the following axioms

$$\begin{cases} (\varphi \bullet \mu) \bullet \mu = 0, \\ (\mu \bullet \varphi) \bullet \mu = 0, \\ \mu \bullet (\varphi \bullet \mu) = 0. \end{cases}$$

Pre-Lie identity applied to the triple (μ, φ, μ) implies

$$(\mu \bullet \varphi) \bullet \mu - \mu \bullet (\varphi \bullet \mu) = (\mu \bullet \mu) \bullet \varphi - \mu \bullet (\mu \bullet \varphi)$$

so

$$(\mu \bullet \varphi) \bullet \mu = \mu \bullet (\varphi \bullet \mu) - \mu \bullet (\mu \bullet \varphi).$$

If we moreover assume that φ belongs to $\chi^k(V)$ then $\mu \bullet (\mu \bullet \varphi) = 0$.

Theorem 2 *Let*

$$\partial^k : \chi^k(V) \longrightarrow \mathcal{C}^{k+n-1}(V)$$

be the linear map defined by

$$\partial^k \varphi = (-1)^{k-1} \mu \bullet \varphi - \varphi \bullet \mu.$$

Then

1. *The image of ∂^k is included in $\chi^{k+n-1}(V)$.*
2. *We obtain the following identity*

$$\partial^{k+n-1} \circ \partial^k = 0.$$

Proof. Let φ be in $\chi^k(V)$ and consider $\partial^k \varphi$. Then

$$(\partial^k \varphi \bullet \mu) \bullet \mu = (-1)^{k-1} ((\mu \bullet \varphi) \bullet \mu) \bullet \mu - ((\varphi \bullet \mu) \bullet \mu) \bullet \mu = 0,$$

and

$$(\mu \bullet \partial^k \varphi) \bullet \mu = (-1)^{k-1} (\mu \bullet (\mu \bullet \varphi)) \bullet \mu - (\mu \bullet (\varphi \bullet \mu)) \bullet \mu = 0,$$

finally

$$\mu \bullet (\partial^k \varphi \bullet \mu) = (-1)^{k-1} \mu \bullet ((\mu \bullet \varphi) \bullet \mu) - \mu \bullet ((\varphi \bullet \mu) \bullet \mu) = 0.$$

Thus $\partial^k \varphi \in \chi^{k+n-1}(V)$. But

$$\begin{aligned} (\partial^{k+n-1} \circ \partial^k) \varphi &= \partial^{k+n-1} ((-1)^{k-1} \mu \bullet \varphi - \varphi \bullet \mu) \\ &= \mu \bullet (\mu \bullet \varphi) + (-1)^k \mu \bullet (\varphi \bullet \mu) + (-1)^k (\mu \bullet \varphi) \bullet \mu + (\varphi \bullet \mu) \bullet \mu = 0 \end{aligned}$$

so

$$\partial^{k+n-1} \circ \partial^k = 0$$

which proves the result.

Corollary 1 *Considering $\delta_i^j = \partial^{i+j(n-1)}$ we get the following complexes:*

$$\begin{aligned} \chi^0(V) &\xrightarrow{\delta_0^0} \chi^{n-1}(V) \xrightarrow{\delta_0^1} \chi^{2n-2}(V) \longrightarrow \dots \longrightarrow \chi^{k(n-1)}(V) \xrightarrow{\delta_0^k} \chi^{(k+1)(n-1)}(V) \longrightarrow \dots \\ \chi^1(V) &\xrightarrow{\delta_1^0} \chi^n(V) \xrightarrow{\delta_1^1} \chi^{2n-1}(V) \longrightarrow \dots \longrightarrow \chi^{k(n-1)+1}(V) \xrightarrow{\delta_1^k} \chi^{(k+1)(n-1)+1}(V) \longrightarrow \dots \\ &\vdots \\ \chi^i(V) &\xrightarrow{\delta_i^0} \chi^{n-1+i}(V) \xrightarrow{\delta_i^1} \chi^{2n-1+i}(V) \longrightarrow \dots \longrightarrow \chi^{k(n-1)+i}(V) \xrightarrow{\delta_i^k} \chi^{(k+1)(n-1)+i}(V) \longrightarrow \dots \\ &\vdots \\ \chi^{n-2}(V) &\xrightarrow{\delta_{n-2}^0} \chi^{2n-3}(V) \xrightarrow{\delta_{n-2}^1} \chi^{3(n-1)-1}(V) \longrightarrow \dots \longrightarrow \chi^{(1+k)(n-1)-1}(V) \xrightarrow{\delta_{n-2}^k} \chi^{(k+2)(n-1)-1}(V) \longrightarrow \dots \end{aligned}$$

2.3 Remark

Let (V, μ) be an algebra of type $\bullet_{n,n}$. It is unital if there exists $1 \in V$ such that

$$\mu(1, 1, \dots, X) = \mu(1, \dots, X, 1) = \dots = \mu(X, \dots, 1) = X$$

for any $X \in V$. Then for any $f \in \text{End}(V)$ we associate the bilinear map φ_f defined by:

$$\varphi_f(X, Y) = \partial^1 f(1, 1, \dots, X, Y).$$

Similary, for any bilinear application φ , we can associate the trilinear application ψ_φ given by :

$$\psi_\varphi(X, Y, Z) = \partial^2 \varphi(1, 1, \dots, 1, X, Y, Z)$$

and if φ belongs to $\mathcal{C}^k(V)$ or $\chi^k(V)$ we consider ψ_φ belonging to $\mathcal{C}^{k+1}(V)$ or $\chi^{k+1}(V)$ given by

$$\psi_\varphi(X_1, \dots, X_{k+1}) = \partial^k \varphi(1, \dots, 1, X_1, \dots, X_{k+1}).$$

Then we get the sequence

$$\mathcal{C}^1(V) \xrightarrow{\phi_1} \mathcal{C}^2(V) \xrightarrow{\phi_2} \mathcal{C}^3(V) \longrightarrow \dots \longrightarrow \mathcal{C}^k(V) \xrightarrow{\phi_k} \mathcal{C}^{k+1}(V) \dots$$

where $\phi_k \varphi = \psi_\varphi$. Computing $\phi_{k+1} \circ \phi_k$ we get

$$\phi_{k+1}(\phi_k(\varphi)) = (\partial^{k+1}(\partial^k \varphi))(1, \dots, 1, X_1, \dots, X_k) = 0.$$

Thus the previous sequence is a complex and we get:

$$\begin{array}{ccccccc} & & \downarrow \phi_{n-2} & & \downarrow \phi_{k(n-1)-1} & & \\ \mathcal{C}^0(V) & \xrightarrow{\delta_0^0} & \mathcal{C}^{n-1}(V) & \xrightarrow{\delta_0^1} \dots \longrightarrow & \mathcal{C}^{k(n-1)}(V) & \xrightarrow{\delta_0^k} \dots & \\ \downarrow \phi_0 & & \downarrow \phi_{n-1} & & \downarrow \phi_{k(n-1)} & & \\ \mathcal{C}^1(V) & \xrightarrow{\delta_1^0} & \mathcal{C}^n(V) & \xrightarrow{\delta_1^1} \dots \longrightarrow & \mathcal{C}^{k(n-1)+1}(V) & \xrightarrow{\delta_1^k} \dots & \\ \downarrow \phi_1 & & \downarrow \phi_{n-1+1} & & \downarrow \phi_{k(n-1)+1} & & \\ \vdots & & \vdots & & \vdots & & \\ \downarrow \phi_{i-1} & & \downarrow \phi_{n-1+i-1} & & \downarrow \phi_{k(n-1)+i-1} & & \\ \mathcal{C}^i(V) & \xrightarrow{\delta_i^0} & \mathcal{C}^{n-1+i}(V) & \xrightarrow{\delta_i^1} \dots \longrightarrow & \mathcal{C}^{k(n-1)+i}(V) & \xrightarrow{\delta_i^k} \dots & \\ \downarrow \phi_i & & \downarrow \phi_{n-1+i} & & \downarrow \phi_{k(n-1)+i} & & \\ \vdots & & \vdots & & \vdots & & \\ \downarrow \phi_{n-3} & & \downarrow \phi_{n-1+n-3} & & \downarrow \phi_{(1+k)(n-1)-2} & & \\ \mathcal{C}^{n-2}(V) & \xrightarrow{\delta_{n-2}^0} & \mathcal{C}^{2n-3}(V) & \xrightarrow{\delta_{n-2}^1} \dots \longrightarrow & \mathcal{C}^{(1+k)(n-1)-1}(V) & \xrightarrow{\delta_{n-2}^k} \dots & \\ \downarrow \phi_{n-2} & & \downarrow \phi_{2n-3} & & \downarrow \phi_{(1+k)(n-1)-1} & & \\ \mathcal{C}^{n-1}(V) & \xrightarrow{\delta_0^1} & \mathcal{C}^{2(n-1)}(V) & \xrightarrow{\delta_0^2} \dots \longrightarrow & \mathcal{C}^{(k+1)(n-1)}(V) & \xrightarrow{\delta_0^{k+1}} \dots & \end{array}$$

3 Deformations and cohomology

Let $A = (V, \mu)$ be a n -ary partially associative algebra. By a deformation of (V, μ) we mean a $\mathbb{K}[[t]]$ - n -ary partially associative algebra $A_t = (V_t, \mu_t)$ where $V_t = V \otimes \mathbb{K}[[t]]$ and $A_t/(tA_t) \simeq A$.

We know that there exists always a cohomology theory which controls these deformations. We recall the construction. Let M be the variety of structure constants and $\mathbb{K}[M]$ the affine coordinate ring of M . We construct a resolution $(\Lambda(X)_*, d) \rightarrow (\mathbb{K}[M], d=0)$ where X is a graded vector space $X = \bigoplus_{i \geq 0} X_i$ and $\Lambda(X)_*$ a graded commutative algebra on X , the differential d satisfying $d(X_i) \subset \Lambda(X_{i-1})$ and $H_i(\Lambda(X), d) = 0$ for $i > 1$. If $L^* = \text{Der}(\Lambda(X)_*, \Lambda(X)_*)$ and δ the differential on L^* induced by d , then $H^*(L, \delta)$ controls the deformations. But this cohomology is too general to be useful for practical computations.

In our case, taking for any $\varphi \in \mathcal{C}^k(V)$,

$$\delta^k \varphi = (-1)^{k-1} \mu \bullet_{2,k} \varphi - \varphi \bullet_{k,2} \mu,$$

we have a complex

$$(\overline{\chi}^k(V) = \chi^k(V) \oplus \text{Ker } \delta^k, \delta^k)$$

such that $H^2(\overline{\chi}^*, \overline{\chi}^*)$ controls the deformations.

4 Graded version of Gerstenhaber's products and associated n -ary algebras

4.1 A relation of degree 7

In this section we claim that n is a natural odd number. In this case we already know that, for a cochain $\phi \in \mathcal{C}^k(V)$, the identity of n -ary algebra (V, μ)

$$(\phi \bullet_{k,n} \mu) \bullet_{k+n-1,n} \mu = 0$$

is not always fulfilled (contrary to the even case) and that we must impose that the cochains satisfy this identity to define a cohomology. As $\mu \bullet \mu = 0$, this identity is equivalent to:

$$\phi \circ \theta_k(\mu) = \phi \circ \sum_{\substack{0 \leq p \leq k-2 \\ 0 \leq q \leq k-2-p}} \text{Id}_p \otimes \mu \otimes \text{Id}_q \otimes \mu \otimes \text{Id}_{k-p-q-2} = 0.$$

In particular, as $\mu \bullet \mu = 0$, we get

$$\mu \circ \theta_k(\mu) = 0.$$

This identity for $n = 3$ writes

$$\begin{aligned} & \mu \circ (\text{Id}_1 \otimes \mu \otimes \text{Id}_1) \circ (\mu \otimes \text{Id}_4) + \mu \circ (\text{Id}_2 \otimes \mu) \circ (\mu \otimes \text{Id}_4) + \mu \circ (\text{Id}_2 \otimes \mu) \circ (\text{Id}_1 \otimes \mu \otimes \text{Id}_3) \\ & + \mu \circ (\mu \otimes \text{Id}_2) \circ (\text{Id}_3 \otimes \mu \otimes \text{Id}_1) + \mu \circ (\mu \otimes \text{Id}_2) \circ (\text{Id}_4 \otimes \mu) + \mu \circ (\text{Id}_1 \otimes \mu \otimes \text{Id}_1) \circ (\text{Id}_4 \otimes \mu) = 0. \end{aligned}$$

so we get

$$\begin{aligned} & (\mu \circ (\text{Id}_1 \otimes \mu \otimes \text{Id}_1) \circ (\mu \otimes \text{Id}_4) + \mu \circ (\mu \otimes \text{Id}_2) \circ (\text{Id}_3 \otimes \mu \otimes \text{Id}_1)) \\ & + (\mu \circ (\text{Id}_2 \otimes \mu) \circ (\mu \otimes \text{Id}_4) + \mu \circ (\mu \otimes \text{Id}_2) \circ (\text{Id}_4 \otimes \mu)) \\ & + (\mu \circ (\text{Id}_2 \otimes \mu) \circ (\text{Id}_1 \otimes \mu \otimes \text{Id}_3) + \mu \circ (\text{Id}_1 \otimes \mu \otimes \text{Id}_1) \circ (\text{Id}_4 \otimes \mu)) = 0. \end{aligned}$$

Similary the identity $\mu \bullet (\phi \bullet \mu) = 0$ is equivalent to:

$$\begin{aligned} & \mu \circ (Id_1 \otimes \phi \otimes Id_1) \circ (Id_1 \otimes \mu \otimes Id_3 + Id_2 \otimes \mu \otimes Id_2 + Id_3 \otimes \mu \otimes Id_1) \\ & + \mu \circ (\phi \otimes Id_2) \circ (\mu \otimes Id_4 + Id \otimes \mu \otimes Id_3 + Id_2 \otimes \mu \otimes Id_2) \\ & + \mu \circ (Id_2 \otimes \phi) \circ (Id_4 \otimes \mu + Id_3 \otimes \mu \otimes Id_1 + Id_2 \otimes \mu \otimes Id_2) = 0 \end{aligned}$$

and for $\phi = \mu$ this identity is fulfilled. It writes

$$\mu \circ (\phi \otimes Id_2) \circ (\mu \otimes Id_4 - Id_3 \otimes \mu \otimes Id_1) + \mu \circ (Id_2 \otimes \phi) \circ (Id_4 \otimes \mu - Id_1 \otimes \mu \otimes Id_3) = 0.$$

Then we get

Proposition 1 *Let (V, μ) be a ternary algebra with multiplication of type $\bullet_{3,3}$. Then μ satisfies the following relations of degree 7:*

1°)

$$\begin{aligned} & \mu \circ (Id_1 \otimes \mu \otimes Id_1) \circ (\mu \otimes Id_4) + \mu \circ (\mu \otimes Id_2) \circ (Id_3 \otimes \mu \otimes Id_1) + \mu \circ (Id_2 \otimes \mu) \circ (\mu \otimes Id_4) \\ & + \mu \circ (\mu \otimes Id_2) \circ (Id_4 \otimes \mu) + \mu \circ (Id_2 \otimes \mu) \circ (Id_1 \otimes \mu \otimes Id_3) \mu \circ (Id_1 \otimes \mu \otimes Id_1) \circ (Id_4 \otimes \mu) = 0. \end{aligned}$$

2°)

$$\mu \circ (\mu \otimes Id_2) \circ (\mu \otimes Id_4 - Id_3 \otimes \mu \otimes Id_1) + \mu \circ (Id_2 \otimes \mu) \circ (Id_4 \otimes \mu - Id_1 \otimes \mu \otimes Id_3) = 0.$$

The interpretation of the first relation show the necessity to distinguish the order of multiplications. A classical meethod consists in grading the initail space so we get:

$$(I) \quad \begin{cases} \mu \circ (Id_1 \otimes \mu \otimes Id_1) \circ (\mu \otimes Id_4) = -\mu \circ (\mu \otimes Id_2) \circ (Id_3 \otimes \mu \otimes Id_1) \\ \mu \circ (Id_2 \otimes \mu) \circ (\mu \otimes Id_4) = -\mu \circ (\mu \otimes Id_2) \circ (Id_4 \otimes \mu) \\ \mu \circ (Id_2 \otimes \mu) \circ (Id_1 \otimes \mu \otimes Id_3) = -\mu \circ (Id_1 \otimes \mu \otimes Id_1) \circ (Id_4 \otimes \mu). \end{cases}$$

We will shortly develop this approach in the following.

4.2 Graded identities

For any two maps $f \in \mathcal{C}^k(V)$ and $g \in \mathcal{C}^l(V)$ we consider

$$f \bullet_i g(X_1, \dots, X_{k+l-1}) = f(X_1, \dots, X_{i-1}, g(X_i, \dots, X_{i+l-1}), \dots, X_{k+l-1})$$

so

$$f \bullet_{k,l} g = \sum_{i=1}^k (-1)^{(i-1)(l-1)} f \bullet_i g$$

We will now work in a \mathbb{Z} -graded vector space $V = \oplus_{n \in \mathbb{Z}} V_n$. We define the suspension (resp. desuspension) of V by $\uparrow V$ (resp. $\downarrow V$), i.e. the graded \mathbb{Z} -graded vector space $\uparrow V = \oplus_{n \in \mathbb{Z}} (\uparrow V)_n$ (resp. $\downarrow V = \oplus_{n \in \mathbb{Z}} (\downarrow V)_n$) with $(\uparrow V)_n = V_{n+1}$ (resp. $(\downarrow V)_n = V_{n-1}$). So the corresponding degree +1 map $\uparrow: V \longrightarrow \uparrow V$ sends $v \in V$ into its suspended copy $\uparrow v \in \uparrow V$, assigns to V the graded vector space $\uparrow V$ and satisfies

$$\uparrow \circ \downarrow = \downarrow \circ \uparrow = Id.$$

More generally we have

$$\uparrow^{\otimes l} \circ \downarrow^{\otimes l} = \downarrow^{\otimes l} \circ \uparrow^{\otimes l} = (-1)^{l(l-1)/2} Id$$

Suppose that the algebra V is graded. If $f : V^{\otimes k} \longrightarrow V$ has a degree $|f|$, then if

$$\phi(f) = \uparrow \circ f \circ \downarrow^{\otimes k}$$

we get

$$\phi(f) \bullet_i \phi(g) = (-1)^{(|g|+k-1)(l-i)+|g|(i-1)} \phi(f \bullet_i g)$$

for graded $f \in C^k(A)$ and $g \in C^l(A)$. Let $\mu \in C^n(A)$ be an application of degree $n-2$. We get

$$\phi(\mu) \bullet_i \phi(\mu) = (-1)^{(n-2+n-1)(n-i)+(n-2)(i-1)} \phi(\mu \bullet_i \mu) = (-1)^{i(n+1)} \phi(\mu \bullet_i \mu).$$

Thus

$$\begin{aligned} \phi(\mu \bullet_{n,n} \mu) &= \phi\left(\sum_{i=1}^n (-1)^{(i-1)(n-1)} \mu \bullet_i \mu\right) = \sum_{i=1}^n (-1)^{(i-1)(n-1)} \phi(\mu \bullet_i \mu) \\ &= \sum_{i=1}^{n-1} (-1)^{(n-1)} \phi(\mu) \bullet_i \phi(\mu) \\ &= (-1)^{(n-1)} \sum_{i=1}^{n-1} \phi(\mu) \bullet_i \phi(\mu). \end{aligned}$$

For example for $n=3$, the graded identity $\mu \bullet_{3,3} \mu$ writes

$$\phi(\mu \bullet_{3,3} \mu) = \sum \phi(\mu) \bullet_i \phi(\mu)$$

et pour $n=2$

$$\phi(\mu \bullet_{2,2} \mu) = -\phi(\mu) \bullet_1 \phi(\mu) - \phi(\mu) \bullet_2 \phi(\mu)$$

All these identities are sign constant. In particular:

Theorem 3 *Let $V = \oplus_{n \in \mathbb{Z}} V_n$ be a \mathbb{Z} -graded vector space. A graded application μ with degree $n-2$ is a Gerstenhaber multiplication of type $\bullet_{n,n}$ if and only if*

$$\sum \phi(\mu) \bullet_i \phi(\mu) = 0.$$

4.3 Composition relations

In the nongraded case we have:

$$\begin{cases} (\mu \bullet_j \mu) \bullet_i \mu = (\mu \bullet_i \mu) \bullet_{j+n-1} \mu & \text{if } i+1 \leq 2n-1 \\ (\mu \bullet_j \mu) \bullet_i \mu = (\mu \bullet_{i+n-1} \mu) \bullet_j \mu & \text{if } 1 \leq j \leq i-n \text{ and } i \geq n+1 \end{cases}$$

If μ is graded with degree $|\mu|$, the commutative rules come from Koszul signs conventions

$$\begin{cases} (\mu \bullet_j \mu) \bullet_i \mu = (-1)^{|\mu||\mu|} (\mu \bullet_i \mu) \bullet_{j+n-1} \mu & \text{if } i+1 \leq 2n-1, \\ (\mu \bullet_j \mu) \bullet_i \mu = (-1)^{|\mu||\mu|} (\mu \bullet_{i+n-1} \mu) \bullet_j \mu & \text{if } 1 \leq j \leq i-n \text{ et } i \geq n+1. \end{cases}$$

Fundamental examples.

- i) For $n=2$, μ is of degree 0 and we obtain the relations of the non graded case.
- ii) For $n=3$, μ is of degree 1 and we get the relations

$$\begin{cases} (\mu \bullet_2 \mu) \bullet_1 \mu = -(\mu \bullet_1 \mu) \bullet_4 \mu, \\ (\mu \bullet_3 \mu) \bullet_1 \mu = -(\mu \bullet_1 \mu) \bullet_5 \mu, \\ (\mu \bullet_3 \mu) \bullet_2 \mu = -(\mu \bullet_2 \mu) \bullet_5 \mu. \end{cases}$$

This gives us the relations claimed in (I)

4.4 On the cohomology in the graded case for $n = 3$

Let $\uparrow A$ be the suspension of the graded space A . Consider μ as an application of degree 1

$$\mu : (\uparrow A)^{\otimes 3} \longrightarrow (\uparrow A)$$

Proposition 2 *If μ is a 3-ary multiplication of degree 1, any $\varphi \in \mathcal{C}^n(\uparrow A)$ satisfies*

$$(\varphi \bullet \mu) \bullet \mu = 0.$$

Consequence.

Let $\delta : \mathcal{C}^n(\uparrow A) \longrightarrow \mathcal{C}^{n+2}(\uparrow A)$ be the 1 degree operation defined by

$$\delta\varphi = \mu \bullet \varphi - (-1)^{|\varphi|} \varphi \bullet \mu$$

where $|\varphi|$ is the degree of φ .

Lemme 3 *(Graded pre-Lie identity)*

Let φ_1 be in $\mathcal{C}^n(\uparrow A)$, φ_2 in $\mathcal{C}^m(\uparrow A)$ and φ_3 in $\mathcal{C}^p(\uparrow A)$. They satisfy

$$(\varphi_1 \bullet \varphi_2) \bullet \varphi_3 - \varphi_1 \bullet (\varphi_2 \bullet \varphi_3) = (-1)^{(m-1)(p-1)} (-1)^{|\varphi_2||\varphi_3|} ((\varphi_1 \bullet \varphi_3) \bullet \varphi_2 - \varphi_1 \bullet (\varphi_3 \bullet \varphi_2))$$

We deduce

$$(\mu \bullet \mu) \bullet \varphi - \mu \bullet (\mu \bullet \varphi) = (-1)^{|\varphi|} ((\mu \bullet \varphi) \bullet \mu - \mu \bullet (\varphi \bullet \mu))$$

and

$$\delta(\delta\varphi) = 0$$

Proposition 3 *The operator $\delta : \mathcal{C}^n(\uparrow A) \longrightarrow \mathcal{C}^{n+2}(\uparrow A)$ defined by*

$$\delta\varphi = \mu \bullet \varphi - (-1)^{|\varphi|} \varphi \bullet \mu$$

gives the complex

$$\mathcal{C}^0(\uparrow A) \longrightarrow \mathcal{C}^3(\uparrow A) \longrightarrow \dots$$

We denote $H^*(\uparrow A, \delta\mu)$ the associated cohomology.

Remark. In [M,R] we give an explanation in operadic terms as the underlying quadratic operad are not Koszul with the usual definition of the cohomology $H^*(\uparrow A, \delta\mu)$.

5 The free algebra $L(V, \bullet_{3,3})$

Let V be a \mathbb{K} -vector space. For even k , the free algebras $L(V, \bullet_{k,k})$ have been described in [3]. But the odd case behaves in a completely different way, as we have already seen it for the cohomology. We are going to describe in detail the case $k = 3$ that is, the case of a 3-ary algebra V with multiplication $\bullet_{3,3}$.

As we have a 3 order product, the free algebra is graded as follows

$$L(V, \bullet_{3,3}) = \oplus_{p \geq 1} L^{2p+1}(V)$$

with

$$L^1(V) = V, \quad L^3(V) = V^{\otimes 3}.$$

Let's describe the further terms.

$$L^5(V) = ((V^{\otimes 3} \otimes V^{\otimes 2}) \oplus (V \otimes V^{\otimes 3} \otimes V) \oplus (V^{\otimes 2} \otimes V^{\otimes 3}))/R_5$$

where R_5 is the sub-space of $(V^{\otimes 3} \otimes V^{\otimes 2}) \oplus (V \otimes V^{\otimes 3} \otimes V) \oplus (V^{\otimes 2} \otimes V^{\otimes 3})$ of relations spanned with vectors which write

$$(v_1 \otimes v_2 \otimes v_3) \otimes v_4 \otimes v_5 + v_1 \otimes (v_2 \otimes v_3 \otimes v_4) \otimes v_5 + v_1 \otimes v_2 \otimes (v_3 \otimes v_4 \otimes v_5).$$

If V is n -dimensional $\dim L^5(V) = 2n^5$.

To describe the other components we denote by $D(k, 3)$, for any positif odd k , the set of triples (a, b, c) satisfying

$$\begin{cases} a, b, c \in \text{odd positif integers,} \\ a + b + c = k. \end{cases}$$

We also need to use a simplified notation for vectors replacing a term $v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_p}$ by $i_1 i_2 \dots i_p$. For example the vector $(v_1 \otimes v_2 \otimes v_3) \otimes v_4 \otimes v_5$ writes $(1 \cdot 2 \cdot 3) \cdot 4 \cdot 5$. We now consider

$$L^7(V) = (\oplus_{(a,b,c) \in D(7,3)} L^a(V) \otimes L^b(V) \otimes L^c(V))/R_7$$

where R_7 is the sub-space of $\oplus_{(a,b,c) \in D(7,3)} L^a(V) \otimes L^b(V) \otimes L^c(V)$ spanned with the vectors

$$\begin{cases} ((1 \cdot 2 \cdot 3) \cdot 4 \cdot 5) \cdot 6 \cdot 7 + (1 \cdot 2 \cdot 3) \cdot (4 \cdot 5 \cdot 6) \cdot 7 + (1 \cdot 2 \cdot 3) \cdot 4 \cdot (5 \cdot 6 \cdot 7), \\ (1 \cdot (2 \cdot 3 \cdot 4) \cdot 5) \cdot 6 \cdot 7 + 1 \cdot ((2 \cdot 3 \cdot 4) \cdot 5 \cdot 6) \cdot 7 + 1 \cdot (2 \cdot 3 \cdot 4) \cdot (5 \cdot 6 \cdot 7), \\ (1 \cdot 2 \cdot (3 \cdot 4 \cdot 5)) \cdot 6 \cdot 7 + 1 \cdot (2 \cdot (3 \cdot 4 \cdot 5) \cdot 6) \cdot 7 + 1 \cdot 2 \cdot ((3 \cdot 4 \cdot 5) \cdot 6 \cdot 7), \\ (1 \cdot 2 \cdot 3) \cdot (4 \cdot 5 \cdot 6) \cdot 7 + 1 \cdot ((2 \cdot 3 \cdot (4 \cdot 5 \cdot 6)) \cdot 7 + 1 \cdot 2 \cdot (3 \cdot (4 \cdot 5 \cdot 6) \cdot 7), \\ ((1 \cdot 2 \cdot 3) \cdot 4 \cdot (5 \cdot 6 \cdot 7) + 1 \cdot (2 \cdot 3 \cdot 4) \cdot (5 \cdot 6 \cdot 7) + 1 \cdot 2 \cdot (3 \cdot 4 \cdot (5 \cdot 6 \cdot 7))). \end{cases}$$

As $\dim(\oplus_{(a,b,c) \in D(7,3)} L^a(V) \otimes L^b(V) \otimes L^c(V)) = 3(\dim L^5(V) \times (\dim V)^2 + 3(\dim L^3(V))^2 \times \dim V) = 9n^7$, we deduce that $\dim L^7(V) = 4n^7$. To describe the general case we need a more efficient coding of vectors. An element of $L^{2p+1}(V)$ writes $1 \cdot (2(\dots(\dots)) \cdot 2p + 1)$ with $p - 1$ brackets. As each bracket has to contain 3 elements, we can code an element of $L^{2p+1}(V)$ by the position of the left brackets. For example $1 \cdot 2 \cdot (3 \cdot 4 \cdot (5 \cdot 6 \cdot 7))$ corresponds to $g_{3,5}$ as the left brackets are at the element 3 and 5. We also suppose that we point the brackets from the left to the right, that is, $g_{j_1, \dots, j_{p-1}}$ belongs to L^{2p+1} with $j_1 \leq j_2 \leq \dots \leq j_{p-2} \leq j_{p-1}$. Thus the above elements of L^7 correspond to $g_{1,1}, g_{1,2}, g_{1,3}, g_{1,4}, g_{1,5}, g_{2,2}, g_{2,3}, g_{2,4}, g_{2,5}, g_{3,3}, g_{3,4}$ and $g_{3,5}$ and R_7 are the relations spanned by the vectors

$$\begin{cases} g_{1,1} + g_{1,4} + g_{1,5}, \\ g_{1,2} + g_{2,2} + g_{2,5}, \\ g_{1,3} + g_{2,3} + g_{3,3}, \\ g_{1,4} + g_{2,4} + g_{3,4}, \\ g_{1,5} + g_{2,5} + g_{3,5}, \\ g_{1,1} + g_{1,2} + g_{1,3}, \\ g_{2,2} + g_{2,3} + g_{2,4}, \\ g_{3,3} + g_{3,4} + g_{3,5}. \end{cases}$$

This coding allows to define the sub-space of relations in any degree by an inductive way. An element of L^{2p+1} has the coding $g_{j_1, \dots, j_{p-1}}$ with $1 \leq j_1 \leq 3, j_1 \leq j_2 \leq 5, \dots, j_{p-2} \leq j_{p-1} \leq 2p-1$. Suppose that we have the relations R_{2p-1} . These relations concern the vectors coded by $g_{j_1, \dots, j_{p-2}}$. The relations of R_{2p+1} are obtained by relations of R_{2p-1} using two rules: Suppose we have a relation in R_{2p-1} , that is implying vectors $g_{j_1, \dots, j_{p-2}}$. We have to explain how we get from such a vector $g_{j_1, \dots, j_{p-2}}$ a vector $g_{l_1, \dots, l_{p-1}}$ involved in R_{2p-1}

- Add the index i in front of each $(p-2)$ -uple of the vectors $g_{j_1, \dots, j_{p-2}}$ involved in the relation, with i successively equal to 1, 2 and 3. We replace each index j_l by $j_l + (i-1)$.

For example: $g_{1,4}$ becomes successively $g_{1,1,4}$, $g_{2,2,5}$ and $g_{3,3,6}$.

- For i successively equal to 1, 2, \dots , $2p-1$, add the index i in front of any $(p-2)$ -uple of vector $g_{j_1, \dots, j_{p-2}}$ involved in the relation; if the index j_1 is less than or equal to i , conserve j_1 , otherwise replace j_1 par j_1+2 . And do the same for all further indices. Rearrange the indices to get $1 \leq j_1 \leq 3, j_1 \leq j_2 \leq 5, \dots, j_{p-2} \leq j_{p-1} \leq 2p-1$.

Thus any relations in R_{2p-1} gives $(2p-1)+3=2p+2$ relations in R_{2p+1} . We have then constructed the generating relations of R_{2p+1} .

Example : relations of R_9 . Each of the 8 relations of R_7 gives 10 relations. For example $g_{1,1}+g_{1,2}+g_{1,3}$ gives

$$\left\{ \begin{array}{l} g_{1,1,1} + g_{1,1,2} + g_{1,1,3}, \\ g_{2,2,2} + g_{2,2,3} + g_{2,2,4}, \\ g_{3,3,3} + g_{3,3,4} + g_{3,3,5}, \\ g_{1,1,1} + g_{1,1,4} + g_{1,1,5}, \\ g_{1,1,2} + g_{1,2,2} + g_{1,2,5}, \\ g_{1,1,3} + g_{1,2,3} + g_{1,3,3}, \\ g_{1,1,4} + g_{1,2,4} + g_{1,3,4}, \\ g_{1,1,5} + g_{1,2,5} + g_{1,3,5}, \\ g_{1,1,6} + g_{1,2,6} + g_{1,3,6}, \\ g_{1,1,7} + g_{1,2,7} + g_{1,3,7}. \end{array} \right.$$

We then get 80 relations. We can solve this system directly or using computer. We solved this system using Mathematica and found $\dim R_9 = 20n^9$ (the rank of the system ist 20). Thus $\dim L^9(V) = 5n^9$.

Remark. Contrary to the previous case there exists some trivial homogeneous products. Any element of L^9 is considered as a product of 3 elements, i.e $u \in L^9$, $u = u_a \otimes u_b \otimes u_c$ with $(a, b, c) \in D(9, 3)$ and $u_a \in L^a$, $u_b \in L^b$, $u_c \in L^c$, or more simply, u is of type $(a, b, c) \in D(9, 3)$. All the elements having a factor in L^7 (that is of type $(3, 3, 1), (3, 1, 3), (1, 3, 3)$) are zero. Also all the homogeneous products of type $(5, 3, 1), (3, 5, 1), (1, 5, 3), (1, 3, 5)$ whose elements in 5 elements are of type $(113)31, 3(311)1, 1(113)3$ and $13(311)$ are zero. By elements of type $(113)31$ we mean elements which can be written $(x_{i_1} \otimes x_{i_2} \otimes x_{i_3}) \otimes x_{i_4} \otimes x_5$ with $x_{i_1}, x_{i_2}, x_{i_3} \in V$ and $x_{i_3}, x_{i_4} \in L^3(V)$. At least the elements of type $11(11(113)), 11((311)11), 1(11(113))1, (11(113))11, ((311)11)11$ and (333) are zero. The rule defining these elements is the following: let us consider an element as a product of 3 elements (a, b, c) . Thus the elements containing

- 3 products of L^3 ,

- 2 products of L^3 in the same bracket (for example $(3, 1, 3)$),
- 2 products of L^3 consecutif but in different brackets (for example $1(113)3$),
- only 1 product of L^3 but neighboring to 2 brackets (for example $11((311)11)$)

are all zero. Moreover remark that a basis of L^9 is given by the vectors coded by

$$g_{3,4,4}, g_{3,4,6}, g_{1,2,4}, g_{1,2,2}, g_{1,1,7}.$$

Definition 2 Let V be a vector space. The free algebra of type $\bullet_{3,3}$ on V is the 3-ary algebra $L(V, \bullet_{3,3}) = \bigoplus_{p \geq 1} L^{2p+1}(V)$ with $L^1(V) = V$, $L^3(V) = V^{\otimes 3}$ and

$$L^{2p+1}(V) = (\bigoplus_{(a,b,c) \in D(2p+1,3)} L^a(V) \otimes L^b(V) \otimes L^c(V)) / R_{2p+1}$$

where R_{2p+1} is the sub-space of $\bigoplus_{(a,b,c) \in D(2p+1,3)} L^a(V) \otimes L^b(V) \otimes L^c(V)$ spanned by the vectors $g_{j_1, \dots, j_{p-1}}$ with $1 \leq j_1 \leq 3, j_1 \leq j_2 \leq 5, \dots, j_{p-2} \leq j_{p-1} \leq 2p-1$ and satisfying the relations defined by the above rules.

It is clear that $L(V, \bullet_{3,3}) = \bigoplus_{p \geq 1} L^{2p+1}(V)$ is of type $\bullet_{3,3}$. If a_1, a_2, a_3 are three homogeneous elements, $a_i \in L^{2p_i+1}$, the product is defined by the class of $a_1 \otimes a_2 \otimes a_3$. This algebra satisfies the following property:

Proposition 4 Let \mathcal{A} be a 3-ary algebra of type $\bullet_{3,3}$ and V a vector space. Then any linear map $f : V \rightarrow \mathcal{A}$ can be factorized in a unique morphism of 3-ary algebras

$$F : L(V, \bullet_{3,3}) \rightarrow \mathcal{A}.$$

Proof. If (\mathcal{A}_1, μ_1) and (\mathcal{A}_2, μ_2) are 3-ary algebras, a linear map $g : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a morphism of algebras if

$$\mu_2(g(X), g(Y), g(Z)) = g(\mu_1(X, Y, Z)),$$

for any $X, Y, Z \in \mathcal{A}_1$. Let $f : V \rightarrow \mathcal{A}$ be a linear map. Consider the linear map $F : L(V, \bullet_{3,3}) \rightarrow \mathcal{A}$ defined on homogeneous components of $L^{2p+1}(V)$ by

$$F(g_{j_1, \dots, j_{p-1}} \otimes (v_p \otimes \dots \otimes v_{2p+1})) = g_{j_1, \dots, j_{p-1}} \otimes (f(v_p) \otimes \dots \otimes f(v_{2p+1})),$$

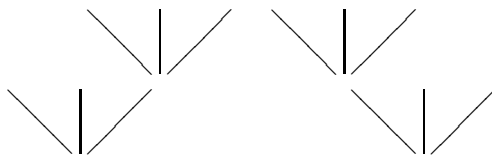
where $g_{j_1, \dots, j_{p-1}} \otimes (v_p \otimes \dots \otimes v_{2p+1})$ corresponds to the vector $(v_{j_1} \otimes v_{j_2} \otimes \dots \otimes v_{j_1} \otimes \dots \otimes v_{j_2} \otimes \dots \otimes v_{j_{p-1}} \otimes v_{j_p} \otimes v_{j_{p+1}}) \dots$. We obtain the expected morphism of algebras.

It remains to give a basis of the free algebra. We have already computed the dimensions of the first homogeneous components. Let us complete these results by describing a basis. For this we use a graphic representation by a planary trees with 3-branching noods (three entries and one exit as the multiplication is 3-ary). We decorate each leave with a basis vector of V to obtain a free family of elements of the free algebra. Suppose V is n -dimensional. Then

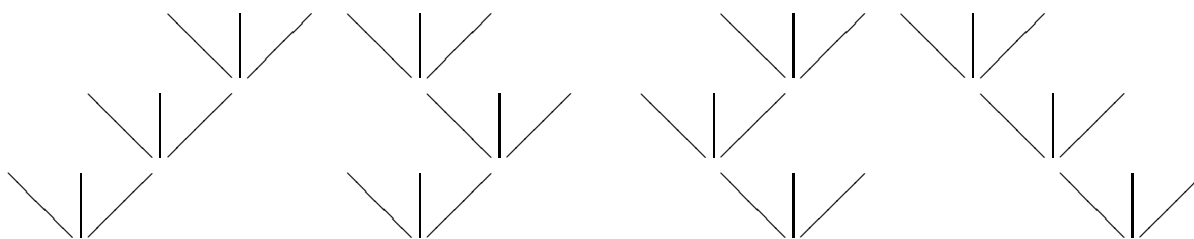
- $\dim L^3(V) = n^3$. A basis is associated to the tree



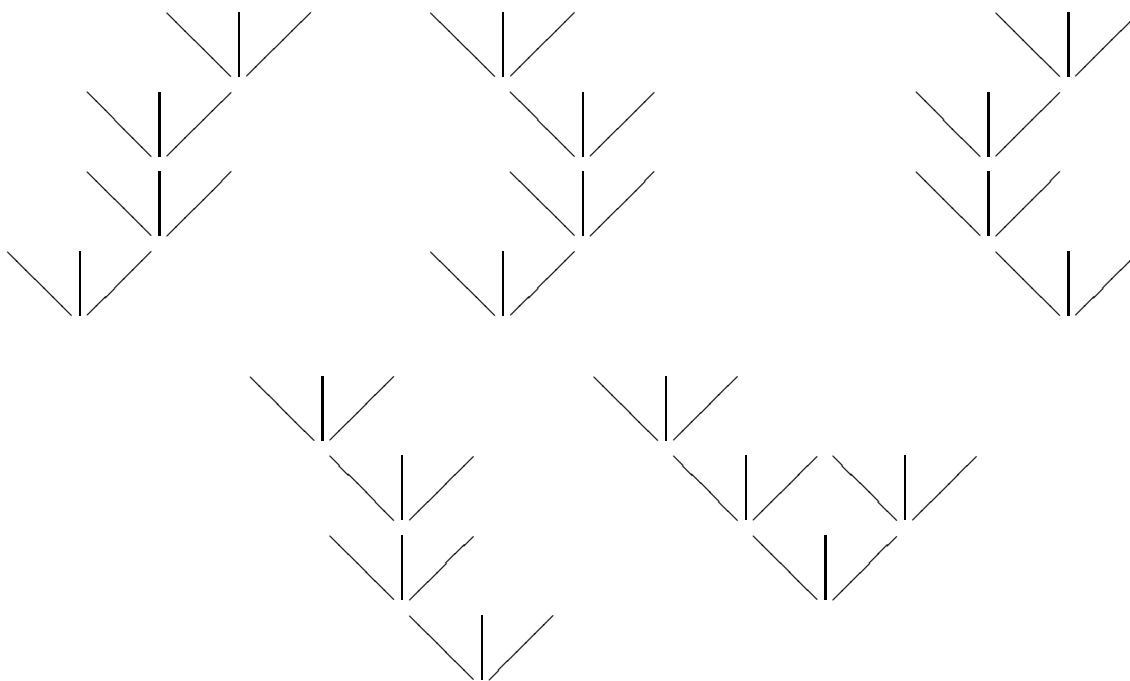
- $\dim L^5(V) = 2n^5$. A basis is associated to the trees



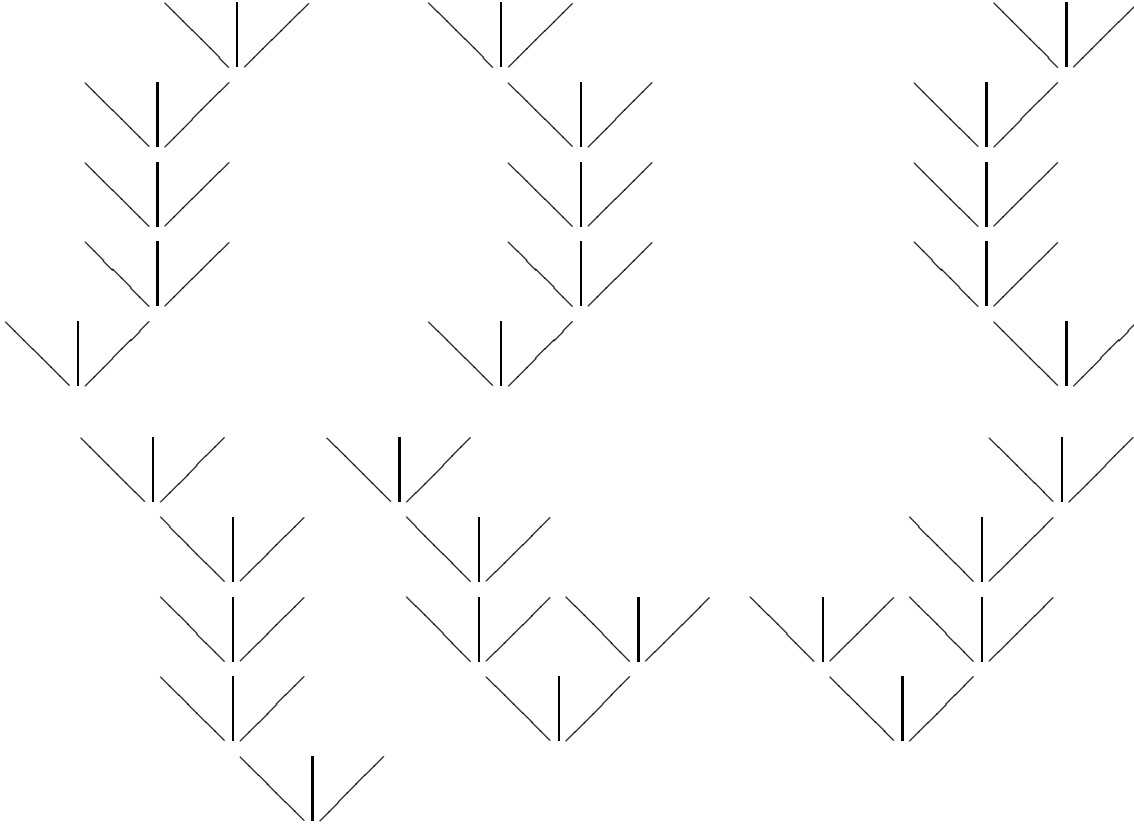
- $\dim L^7(V) = 4n^7$. A basis corresponds to the trees



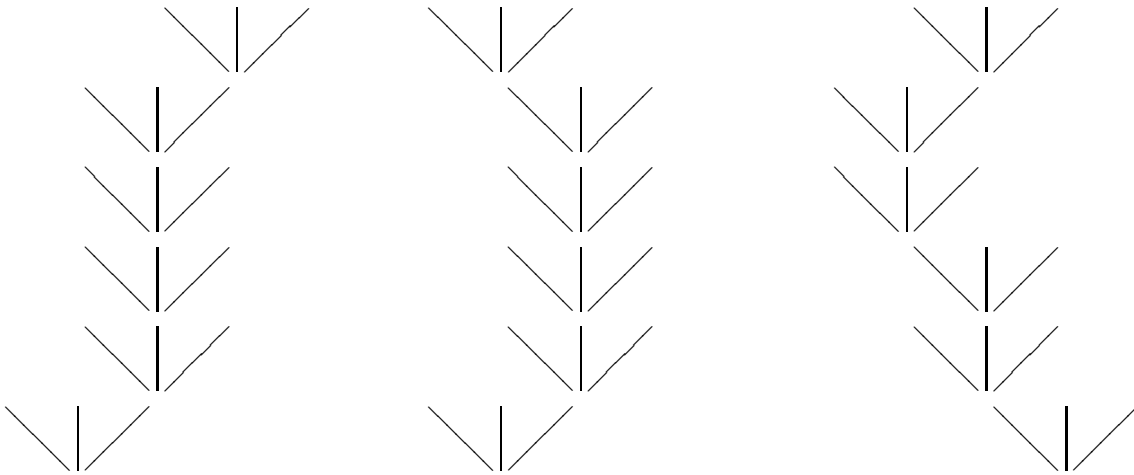
- $\dim L^9(V) = 5n^9$. A basis is given by

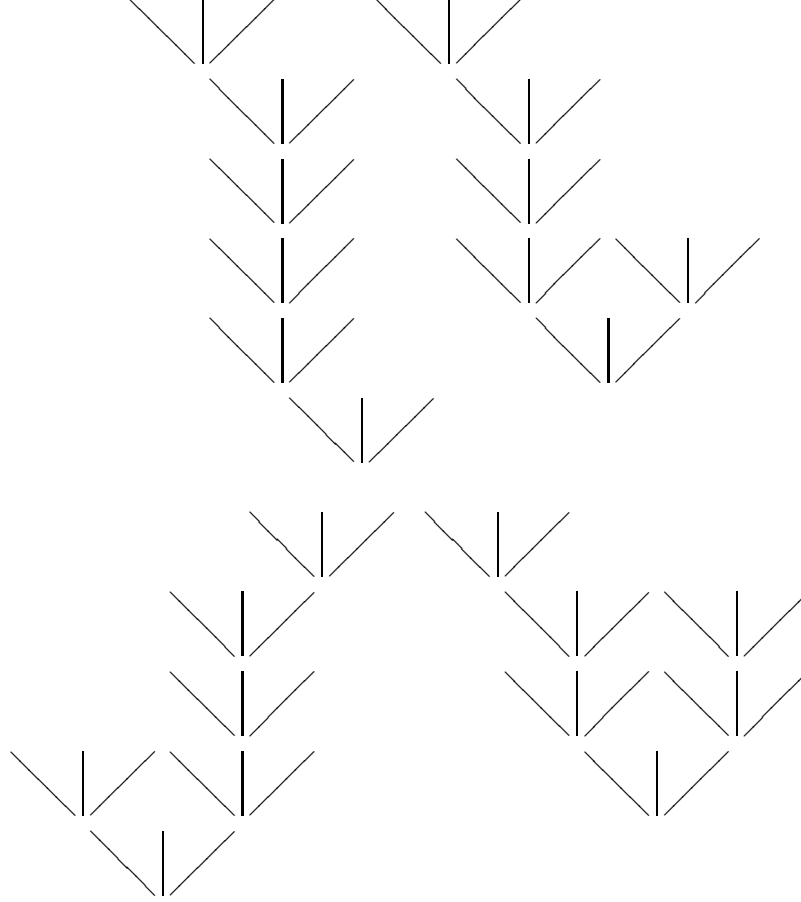


- $\dim L^{11}(V) = 6n^{11}$. A basis is given by



- $\dim L^{13}(V) = 7n^{13}$. A basis is given by

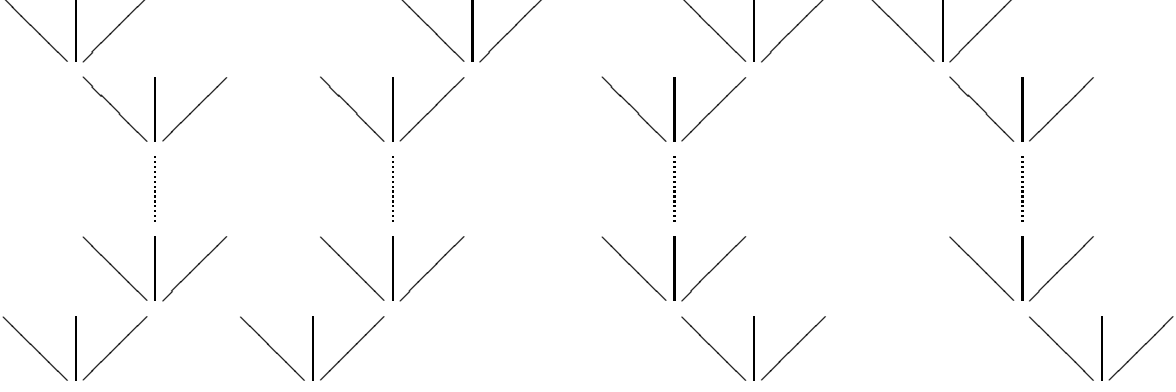




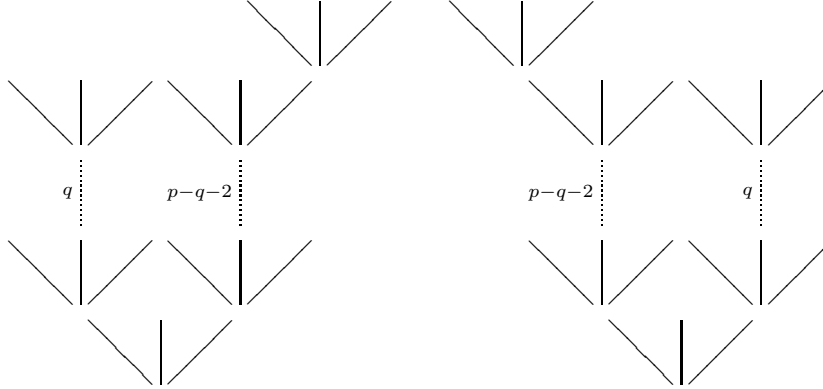
The choice of the basis is non canonical. But we choose them for symmetry reasons. The rules providing the relations of the sub-space R_{2p+1} are easy to implement in order to solve the corresponding linear system. This gives the dimensions of the spaces $L^{2p+1}(V)$ (in fact we find the dimensions of the modules of the associated operad). We have illustrated this approach in small dimensions above. Let us notice that we can however present basic vectors for the relations associated to the elements of

$$L^{2p-1} \otimes L^1 \otimes L^1 \oplus L^1 \otimes L^{2p-1} \otimes L^1 \oplus L^1 \otimes L^1 \otimes L^{2p-1}.$$

These elements correspond to the trees



The other are of the form



where $q = 1, \dots, [\frac{p-2}{2}]$ and $[\cdot]$ indicates the integer part of a rational number. If p is even, the last two trees are related. If p is odd, these trees are independent. We deduce:

Theorem 4 *For any p we have*

$$\dim L^{2p+1}(V) = (p+1)n^{2p+1}$$

where $n = \dim V$.

Remark. Recall that for any vector space V , the associated tensor algebra $T(V)$ is the unique solution, up to isomorphism, of the universal problem which determine from a linear application $f : M \rightarrow A$ in an associative algebra A , a morphism of associative algebra $T(V) \rightarrow A$. The construction of this algebra comes from the isomorphisms

$$\Phi_{n,m} : T^{\otimes n}(V) \otimes T^{\otimes m}(V) \rightarrow T^{\otimes(n+m)}(V)$$

defined by

$$\Phi_{n,m}((x_1 \otimes x_2 \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m)) = x_1 \otimes x_2 \cdots \otimes x_n \otimes y_1 \otimes y_2 \cdots \otimes y_m.$$

In fact the multiplication μ of $T(V)$ is given by

$$\mu((x_1 \otimes x_2 \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m)) = \Phi_{n,m}((x_1 \otimes x_2 \cdots \otimes x_n) \otimes (y_1 \otimes y_2 \cdots \otimes y_m))$$

and the associativity of the multiplication follows from

$$\Phi_{n+m,p} \bullet (\Phi_{n,m} \otimes Id_p) = \Phi_{n+m,p} \bullet (Id_n \otimes \Phi_{m,p}).$$

We can define an other isomorphism non longer adapted to the associative structure but adapted to the n -ary structure. For this we consider the family of vectorial isomorphisms

$$\Psi_{n,m,p} : T^{\otimes n}(V) \otimes T^{\otimes m}(V) \otimes T^{\otimes p}(V) \rightarrow T^{\otimes n+m+p}(V)$$

satisfying

$$\begin{cases} \Psi_{n,m+p+q,r} \bullet (Id_n \otimes \Psi_{m,p,q} \otimes Id_r) &= -2\Psi_{n,m+p+q,r} \bullet (Id_{n+m} \otimes \Psi_{p,q,r}) \\ &= -2\Psi_{n,m+p+q,r} \bullet (\Psi_{n,m,p} \bullet Id_{q+r}). \end{cases}$$

6 Extension of the notion of coassociative algebras for n -ary algebras

For $n = 2$ we have that 2-ary partially associative algebras are just associative algebras and we can define coassociative coalgebras with the wellknown relations between these two structures. In fact, the dual space of a coassociative algebra can be provided with a structure of associative algebra, the dual space of a finite dimensional associative algebra can be provided with a structure of coassociative coalgebra structure and also, if (A, μ) is an associative algebra and (M, Δ) a coassociative coalgebra, the space $Hom(M, A)$ can be provided with an associative algebra structure. All these notions can be extended to n -ary algebras.

A n -ary partially associative algebra has a product μ satisfying Equation (1) written in the following form

$$\sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}) = 0.$$

Then we get the definition of partially coassociative n -ary coalgebra.

Definition 3 A n -ary comultiplication on a \mathbb{K} -vector space M is a map

$$\Delta : M \rightarrow M^{\otimes n}.$$

A n -ary partially coassociative coalgebra is a \mathbb{K} -vector space M provided with a n -ary comultiplication Δ satisfying

$$\sum_{p=0}^{n-1} (-1)^{p(n-1)} (Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta = 0.$$

A n -ary totally coassociative coalgebra is a \mathbb{K} -vector space M provided with a n -ary comultiplication Δ satisfying

$$(Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta = (Id_q \otimes \Delta \otimes Id_{n-1-q}) \circ \Delta,$$

for any $p, q \in \{0, \dots, n-1\}$.

If (\mathcal{A}, μ) is a n -ary algebra and (M, Δ) n -ary coalgebra we denote by

$$A(\mu) = \sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}),$$

$$\tilde{A}(\Delta) = \sum_{p=0}^{n-1} (-1)^{p(n-1)} (Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta.$$

For any natural number n and any \mathbb{K} -vector spaces E and F , we denote by

$$\lambda_n : Hom(E, F)^{\otimes n} \longrightarrow Hom(E^{\otimes n}, F^{\otimes n})$$

the natural embedding

$$\lambda_n(f_1 \otimes \dots \otimes f_n)(x_1 \otimes \dots \otimes x_n) = f_1(x_1) \otimes \dots \otimes f_n(x_n).$$

Proposition 5 *The dual space of a n -ary partially coassociative coalgebra is provided with a structure of n -ary partially associative algebra.*

Proof. Let (M, Δ) be a n -ary partially coassociative coalgebra. We consider the multiplication on the dual vector space M^* of M defined by :

$$\mu = \Delta^* \circ \lambda_n.$$

It provides M^* with a n -ary partially associative algebra structure. In fact we have

$$\mu(f_1 \otimes f_2 \otimes \dots \otimes f_n) = \mu_{\mathbb{K}} \circ \lambda_n(f_1 \otimes f_2 \otimes \dots \otimes f_n) \circ \Delta \quad (3)$$

for all $f_1, \dots, f_n \in M^*$ where $\mu_{\mathbb{K}}$ is the multiplication of \mathbb{K} . Equation (3) becomes :

$$\begin{aligned} & \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p})(f_1 \otimes f_2 \otimes \dots \otimes f_{2n-1}) \\ &= \mu_{\mathbb{K}} \circ (\lambda_n(f_1 \otimes \dots \otimes f_p \otimes \mu(f_{p+1} \otimes \dots \otimes f_{p+n}) \otimes f_{p+n+1} \otimes \dots \otimes f_{2n-1})) \circ \Delta \\ &= \mu_{\mathbb{K}} \circ \lambda_n(f_1 \otimes \dots \otimes f_p \otimes (\mu_{\mathbb{K}} \circ \lambda_n(f_{p+1} \otimes \dots \otimes f_{p+n}) \circ \Delta) \otimes f_{p+n+1} \otimes \dots \otimes f_{2n-1}) \circ \Delta \\ &= \mu_{\mathbb{K}} \circ (Id_p \otimes \mu_{\mathbb{K}} \otimes Id_{n-1-p}) \circ \lambda_{2n-1}(f_1 \otimes \dots \otimes f_{2n-1}) \circ (Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta. \end{aligned}$$

Using associativity and commutativity of the multiplication in \mathbb{K} , we obtain

$$\forall p, q \in \{0, \dots, n-1\}, \quad \mu_{\mathbb{K}} \circ (Id_p \otimes \mu_{\mathbb{K}} \otimes Id_{n-1-p}) = \mu_{\mathbb{K}} \circ (Id_q \otimes \mu_{\mathbb{K}} \otimes Id_{n-1-q}),$$

so

$$\begin{aligned} & \sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}) \\ &= \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes Id_{n-1}) \circ \lambda_{2n-1}(f_1 \otimes \dots \otimes f_{2n-1}) \circ \sum_{p=0}^{n-1} (-1)^{p(n-1)} (Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta = 0 \end{aligned}$$

and (M^*, μ) is a n -ary partially associative algebra. \square

Proposition 6 *The dual vector space of a finite dimensional n -ary partially associative algebra has a n -ary partially associative coalgebra structure.*

Proof. Let \mathcal{A} be a finite dimensional n -ary partially associative algebra and let $\{e_i, i = 1, \dots, n\}$ be a basis of \mathcal{A} . If $\{f_i\}$ is the dual basis then $\{f_{i_1} \otimes \dots \otimes f_{i_n}\}$ is a basis of $(\mathcal{A}^*)^{\otimes n}$. The coproduct Δ on \mathcal{A}^* is defined by

$$\Delta(f) = \sum_{i_1, \dots, i_n} f(\mu(e_{i_1} \otimes \dots \otimes e_{i_n})) f_{i_1} \otimes \dots \otimes f_{i_n}.$$

In particular

$$\Delta(f_k) = \sum_{i_1, \dots, i_n} C_{i_1, \dots, i_n}^k f_{i_1} \otimes \dots \otimes f_{i_n}$$

where C_{i_1, \dots, i_n}^k are the structure constants of μ related to the basis $\{e_i\}$. Then Δ is the comultiplication of a n -ary partially coassociative coalgebra. \square

Now we study the convolution product. Let us recall that if (\mathcal{A}, μ) is associative \mathbb{K} -algebra and (M, Δ) a coassociative \mathbb{K} -coalgebra then the convolution product

$$f \star g = \mu \circ \lambda_2(f \otimes g) \circ \Delta$$

provides $\text{Hom}(M, \mathcal{A})$ with an associative algebra structure. This result can be extended to the n -ary partially associative algebras and partially coassociative coalgebras.

Proposition 7 *Let (\mathcal{A}, μ) be a n -ary partially associative algebra and (M, Δ) a n -ary totally coalgebra. Then the algebra $(\text{Hom}(M, \mathcal{A}), \star)$ is a n -ary partially associative algebra where \star is the convolution product :*

$$f_1 \star f_2 \star \dots \star f_n = \mu \circ \lambda_n(f_1 \otimes f_2 \otimes \dots \otimes f_n) \circ \Delta.$$

Proof. Let us compute the convolution product of functions of $\text{Hom}(M, \mathcal{A})$. We have

$$\begin{aligned} & f_1 \star \dots \star f_{i-1} \star (f_i \star f_{i+1} \star \dots \star f_{i+n-1}) \star f_{i+n} \star \dots \star f_{2n-1} \\ &= \mu \circ \lambda_n(f_1 \otimes f_2 \otimes \dots \otimes f_{i-1} \otimes (f_i \star \dots \star f_{i+n-1}) \otimes f_{i+n} \otimes \dots \otimes f_n) \circ \Delta \\ &= \mu \circ \lambda_n(f_1 \otimes \dots \otimes f_{i-1} \otimes (\mu \circ \lambda_n(f_i \otimes \dots \otimes f_{i+n-1}) \circ \Delta) \otimes f_{i+n} \otimes f_{2n-1}) \circ \Delta \\ &= \mu \circ (Id_{i-1} \otimes \mu \otimes Id_{n-i}) \circ \lambda_{2n-1}(f_1 \otimes f_2 \otimes \dots \otimes f_{2n-1}) \circ (Id_{i-1} \otimes \Delta \otimes Id_{n-i}) \circ \Delta, \end{aligned}$$

As Δ is a n -ary totally associative product, we have

$$\begin{aligned} & A(\star)(f_1 \otimes \dots \otimes f_{2n-1}) \\ &= \sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}) \circ \lambda_{2n-1}(f_1 \otimes \dots \otimes f_{2n-1}) \circ (Id_p \otimes \Delta \otimes Id_{n-1-p}) \circ \Delta \\ &= \left(\sum_{p=0}^{n-1} (-1)^{p(n-1)} \mu \circ (Id_p \otimes \mu \otimes Id_{n-1-p}) \right) \circ \lambda_{2n-1}(f_1 \otimes \dots \otimes f_{2n-1}) \circ (\Delta \otimes Id_{n-1}) \circ \Delta = 0. \end{aligned}$$

7 Some examples of n -ary algebras

1. Let \mathfrak{g} be a Lie algebra. The associator related to the Lie bracket is

$$A(X, Y, Z) = [[X, Y], Z] - [X, [Y, Z]] = [[X, Z], Y].$$

If \mathfrak{g} is a 4 step nilpotent Lie algebra the multiplication $\mu(X, Y, Z) = A(X, Y, Z)$ is 3-ary of type $\bullet_{3,3}$.

2. Let μ a n -ary multiplication of type $\bullet_{n,n}$ on a vector space V . This multiplication is commutative if, for any $v_i \in V$,

$$\sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} \mu(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = 0,$$

where S_n is the n -order symmetric group and $\varepsilon(\sigma)$ is the signature of the element σ of S_n . The 3-ary algebras of the previous examples are commutative. A non-commutative version is based on the Roby algebras. A Roby algebra is constructed in the following way: Let V be a vector space and $T(V)$ its associated tensor algebra. For any integer k , we consider the ideal $I(V, k)$ of $T(V)$ generated by the products of symmetric tensors of length k . The exterior algebra of order k , or Roby algebra of order k , is by definition

$$\Lambda(V, k) = T(V)/I(V, k).$$

For $k = 2$ we get the usual exterior algebra. For $k = 3$, the ideal $I(V, 3)$ is generated by tensors of type

$$\left\{ \begin{array}{l} v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 + v_3 \otimes v_2 \otimes v_1 + v_1 \otimes v_3 \otimes v_2 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2, \\ v_1^{\otimes 2} \otimes v_2 + v_2 \otimes v_1^{\otimes 2}, \\ v_1 \otimes v_2^{\otimes 2} + v_2^{\otimes 2} \otimes v_1, \end{array} \right.$$

with distinct vectors v_1, v_2, v_3 . If μ is the multiplication in $\Lambda(V, 3)$, it satisfies

$$\left\{ \begin{array}{l} \mu(v_1, v_2, v_3) + \mu(v_2, v_1, v_3) + \mu(v_3, v_2, v_1) + \mu(v_1, v_3, v_2) + \mu(v_2, v_3, v_1) + \mu(v_3, v_1, v_2) = 0 \\ \mu(v_1, v_1, v_2) + \mu(v_2, v_1, v_1) = 0. \end{array} \right.$$

with distinct vector v_1, v_2, v_3 . We deduce $\mu(v_1, v_1, v_1) = 0$. If we now claim that μ is a multiplication of type $\bullet_{3,3}$, such algebra is its exterior version.

3. A Poisson algebra of type $\bullet_{3,3}$ can be defined as a commutative algebra (V, μ) of type $\bullet_{3,3}$ with a Lie bracket satisfying

$$[\mu(X, Y, Z), T] = \mu([X, T], Y, Z) + \mu(X, [Y, T], Z) + \mu(X, Y, [Z, T])$$

for any $X, Y, Z, T \in V$. If V is a \mathbb{Z}_2 -graded vector space, we consider on $V = V_0 \oplus V_1$ a graded Lie bracket which provides V with a super Lie algebra structure. Thus this bracket satisfies

$$\left\{ \begin{array}{lcl} [X_1, X_2] & = & -[X_2, X_1] \\ [X_1, Y_2] & = & -[Y_2, X_1] \\ [Y_1, Y_2] & = & [Y_2, Y_1] \end{array} \right.$$

for any $X_1, X_2 \in V_0$ and $Y_1, Y_2 \in V_1$. It also satisfies the graded Jacobi identity. A superalgebra Poisson structure of type $\bullet_{3,3}$ on $V = V_0 \oplus V_1$ is given by a multiplication μ of type $\bullet_{3,3}$ and a graded Lie bracket satisfying

$$[\mu(X, Y, Z), T] = \mu([X, T], Y, Z) + \mu(X, [Y, T], Z) + \mu(X, Y, [Z, T])$$

An example is given by the F -algebras defined in [6] which are some generalisation of the superalgebra associated to the super-symmetry. In fact such algebra (for $F = 3$) is defined on a graded Lie algebra ($V = V_0 \oplus V_1, [,]$) provided with a commutative multiplication of type $\bullet_{3,3}$, denoted $\{, , \}$ in this case, and satisfying

$$\{V_i, V_j, V_k\} = 0$$

as soon as $(i, j, k) \neq (1, 1, 1)$,

$$\{V_1, V_1, V_1\} \subseteq V_0$$

and the graded Leibniz relations

$$[X, \{Y_1, Y_2, Y_3\}] = \{[X, Y_1], Y_2, Y_3\} + \{Y_1, [X, Y_2], Y_3\} + \{Y_1, Y_2, [X, Y_3]\}$$

for any $X \in V_0$ et $Y_1, Y_2, Y_3 \in V_1$,

$$[Y, \{Y_1, Y_2, Y_3\}] + [Y_1, \{Y_2, Y_3, Y\}] + [Y_2, \{Y_3, Y, Y_2\}] + [Y_3, \{Y, Y_1, Y_2\}] = 0$$

for any $Y, Y_1, Y_2, Y_3 \in V_1$.

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